Visualizing Music Using Geometry

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Abstract

Mathematicians have long had an interest in describing music in mathematical terms. Pythagoras (500 BC), Euclid (300 BC), and Euler (1739) all made important contributions to music theory. We look at various ways to visualize music using geometry. We can visualize the relationship between tones as a linear keyboard, a never-ending circle, or a spiral in two or three dimensions. Scales are constructed using geometric series of frequencies, starting with the perfect fifth interval. These scales can be visualized as waveforms or spirals in two or three dimensions. We compare and contrast ancient and medieval tuning systems versus modern tuning systems, and applications of each. Major and minor chords have their own frequency relationships that sometimes conflict with the perfect fifth. Chords can be visualized using a tiling or mesh; these meshes can be adapted to portray different kinds of tunings. We look at advantages and disadvantages of each kind of geometry for visualizing music.

Introduction

The various aspects of relationship between mathematics and music are well known, having been recognized since the days of Pythagoras (500 BC) and Euclid (300 BC). Mathematics can be used to describe tones, scales, harmonic intervals, chords and chord progressions, time sequences, and rhythm. Mathematics was used extensively over a period of 500 years to develop the tuning systems of today. Mathematics is also used to construct modern electronic music instruments, which use combinations of hardware (chips) and software (programming such as frequency analysis) to generate tones (Benson, 2006).

Mathematics is used in music composition in many ways. Various programs allow one to put notes on a music staff for later playback. Data formats and protocols such as MIDI allow one to encode and transfer music as files. These files can also be used to control electronic musical instruments. Composers can use computers to create aleatory (random) music or rule-based music (twelve-tone, serial, or tone row techniques) that formerly had to be computed by hand (Wright, 2009).

Much of the mathematics we used to describe music is algebraic in nature. The relationships between frequencies and rhythms can generally be described as ratios. For example, 2:1 is the ratio of frequencies of tones one octave apart. 4:1 is the ratio of beats to measures in much of music. In this paper, we will look specifically at applications of geometry to visualizing music. So while the algebra still remains, we will see (literally) how these relationships develop.

1. Tones

Let's start with some terminology. In the following discussion, we usually use the term *key* to indicate a *physical key* on the piano keyboard. Key has other meanings, such as a piece of music being in the key of C major or the key of E-flat. But we won't be using the term that way much. The term *note* refers to a name we give to a musical pitch. We frequently talk about notes on a musical score or staff. Sometimes *key* and *note* are used almost interchangeably. For example, Middle C is both a specific key near the middle of the piano keyboard, and also a particular placement of an object on a musical staff.

Figure 1: Notes on the musical staff

The term *tone* refers to the sound made when a musical note is actually played. We describe *tones* in terms of their frequency, amplitude, and timbre. Keys and notes are intrinsically visual, because we can see keys on the piano and we can see notes on the staff. On the other hand tones are primarily auditory. Our goal here is to describe these auditory phenomena in a more visual way.

One common way to visualize tones is in terms of a piano keyboard. The relationship between tones looks linear, even though the frequencies themselves are not linear. This visualization allows us to see all the notes in the scale. We don't need to look at the whole piano keyboard. The piano keys repeat in a pattern called *octaves*. The word octave indicates that there are eight keys. Each octave looks the same as the next and the previous. But some octaves (more to the left of the piano keyboard) have lower tones, and some (more to the right) have higher tones. So we have a natural relationship between lower tones (leftward) and higher tones (rightward).

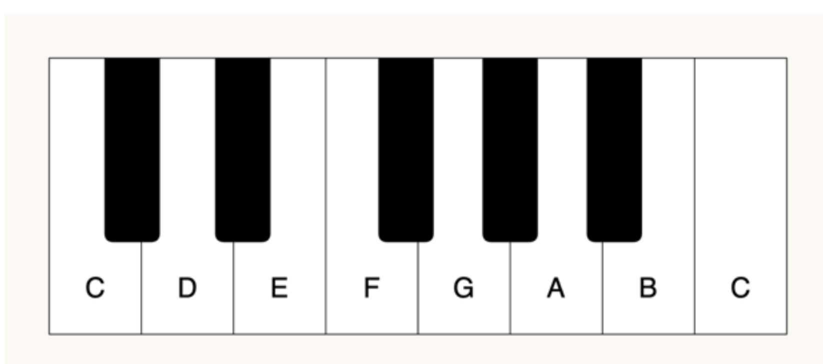

Figure 2: Portion of piano keyboard

Figure 2 shows a portion of a piano keyboard. There are eight keys, from C up to the next C. The first C (on the left) could be any C on the piano keyboard, but maybe we can assume the first C is Middle C. Then the A would have a frequency of 440 Hertz. You've probably heard of A 440. If you play all the notes from C up to the next C, you've played the C major scale. Once you've got to the C on the right hand edge, there are usually more piano keys (unless you got to the very end of the piano keyboard). So you could keep playing another octave. For this reason, this static keyboard may not be the best way to visualize 88 tones from the 88 keys on a piano keyboard.

Figure 3 shows a circle that has all the notes from C up to C. Moving clockwise around the circle is equivalent to moving to the right up the piano keyboard (Wright, 2009). We can start on C and play all the notes up to B, then we can keep going. The circle theoretically goes around infinitely, but for a practical reason, we limit the number of octaves. This is not because of any limitation of acoustic or electronic musical instruments, but because of a limitation of human hearing. The generally accepted limits of human hearing are around 20 Hz to 20,000 Hz (Benson, 2006). This represents a distance of about 10 octaves or 120 notes. The piano keyboard has 88 of these notes. The lowest key on the piano keyboard has a frequency of about 21 Hz, and the highest note has a frequency of about 4200 Hz (Benson, 2006).

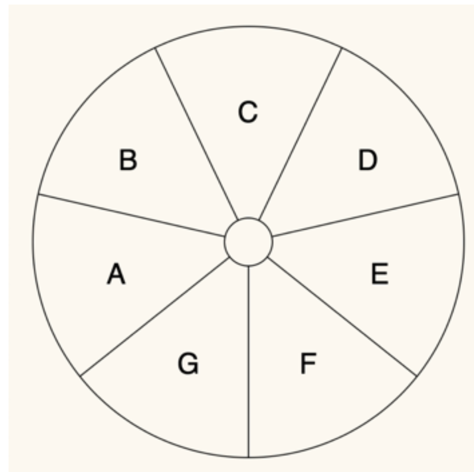

Figure 3: 7 tones in a circle

You may have noticed a few anomalies in the foregoing discussion. We said there are eight keys in the octave, which accounts for its name. But the keyboard illustrated in *Figure 2* actually has more keys. The eight keys only account for what we call the *white keys*. But there are some additional *black keys* in between the white ones. Also, we said that the 10 octave range of human hearing consists of about 120 notes. So really, there are 12 notes in an octave, not just eight. The black keys have names like B-flat (B♭) and F-sharp (F♯). Here is what all 12 notes look like in a circle.

The circle in *Figure 4* shows that we can play not only the white keys, but also the black keys, and they all continue around clockwise (higher) or counterclockwise (lower). A scale that uses only white keys is called a *diatonic* scale. Some scales use only one black key or only a few black keys. For example, the F major scale uses the B♭ black key, and the G major scale uses the F♯ black key. A scale that uses all the black and white keys is called a *chromatic* scale.

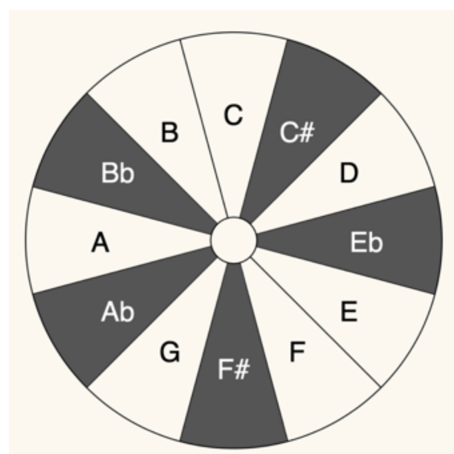

Figure 4: 12 tones in a circle

We might be able to visualize more than just one octave by arranging the keys in a spiral. We could put lower notes on the inner part of the spiral and higher notes on the outer part. To have more than one octave in one chart, we probably want a way to distinguish one octave from another. One way musicians do this is to give each key an octave number as well as a letter (Wright, 2009). For example, C_4 is middle C. A_4 is A 440 because it is in the same octave with middle C. The C below middle C is C_3. The C above middle C is C_5. The highest note on the piano is C_8. The lowest note on the piano is A_0. *Figure 5* shows the piano keyboard with octave numbers.

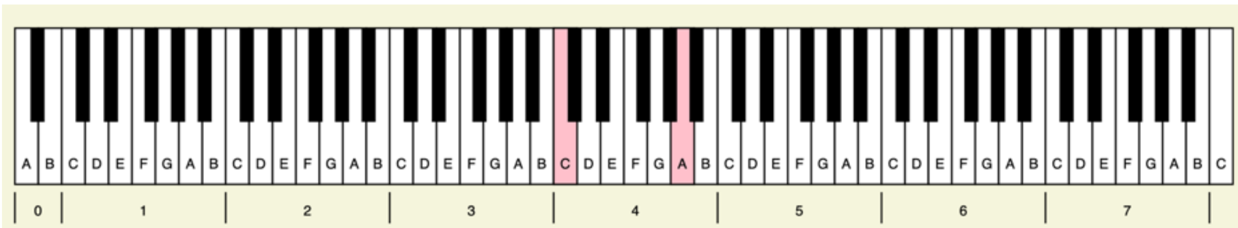

Figure 5: Piano keyboard with C_4 and A_4 highlighted

Figure 6 shows the octaves arranged in a spiral instead of a linear keyboard. Note that in the spiral, all the C notes occupy the same angle around the spiral. All the D notes occupy the same angle. This shows that we can divide the tones into *tone classes*, *note classes*, or *pitch classes* (Benson, 2006, Wright, 2009). All C notes are in the same *tone class*. All D notes are in the same tone class. And so on. The position of each tone relative to the center of the spiral indicates its frequency. Tones farther away from the center have higher frequency and sound higher. This concept of tone classes is like the *congruence classes* you might encounter in number theory or abstract algebra (Benson, 2006) or the *equivalence classes* you might encounter in set theory (Wright, 2009).

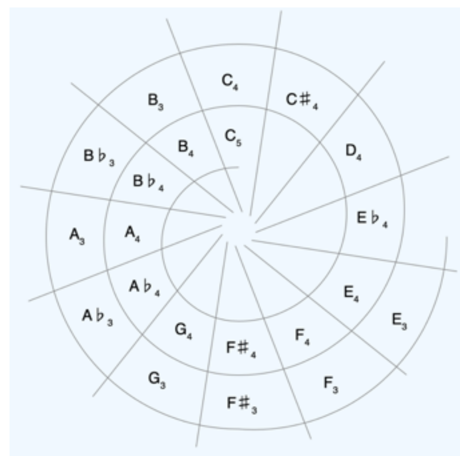

Figure 6: 21 tones in a spiral

It might be more instructive to view this in 3D. In *Figure 7*, the tones spiral around the z axis and tones with higher z values have higher frequencies. Tones in the same tone class have the same colors. Shepard (1982) calls this spiral a *helix* with a *chroma* dimension that goes around the z axis (indicating the different tone classes), and a *height* dimension that goes up the z axis (indicating the octaves). (See *Figure 8*.)

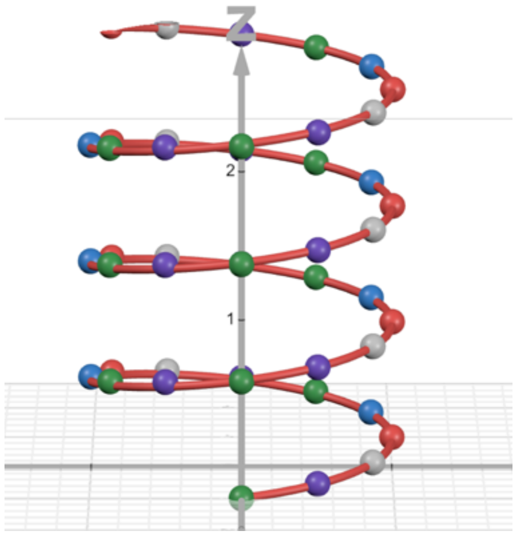

Figure 7: Tones in a 3D spiral

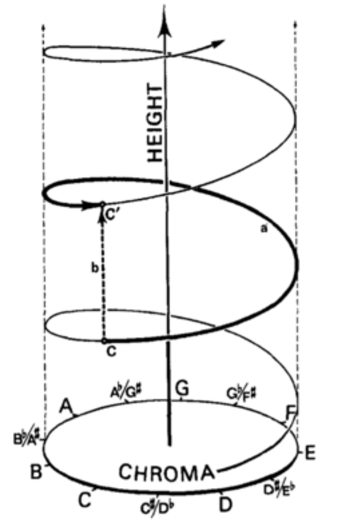

Figure 8: Chroma helix (Shepard, 1982)

The green balls in the front of the drawing could be C tones. Then the purple ones to the right would be C♯, the gray ones D, and the red ones E♭. Tones in the same tone class appear one above another.[1] All tones are equally distant from the z axis, but their z coordinates indicate relative frequencies. In this drawing, each tone rises above the previous and is below the next tone, indicating a change of frequency.

2. Pitches and Scales

Tones that are one octave apart have frequencies in a 2:1 ratio. For example, the A above A 440 has a frequency of 880 Hz. The A below A 440 has a frequency of 220 Hz. Musicians and mathematicians have recognized this for centuries. A string that is one foot long, when bowed or plucked, has a sound that is higher than a string that is two feet long, but otherwise sounds the same. We describe tones in the same tone class as all sounding somewhat the same, even if they are an octave apart. That is how we recognize an octave. The same applies to lengths of pipe when air is supplied and pieces of metal when struck with a hammer.

Musicians recognize tones that are an octave apart as being harmonious; they sound good together. Sometimes, two tones an octave apart might actually sound like just one tone. That is how well they go together.

[1] Unfortunately, there is some duplication of colors in Figure 7. This is due to a limitation of the drawing program, Desmos. It would be better if each tone class had its own distinct color.

Pythagoras discovered that tones whose frequencies are the ratios of small integers sound harmonious (Benson, 2006). The smaller and closer the integers are, the more harmonious the sound. Two tones whose frequency ratio is 2:1 are an octave apart. After the octave, the next most pleasing combination has a frequency ratio of 3:2 (Himpel, 2022). These tones form what is called a *perfect fifth*. In music, a perfect fifth is not a fraction, although the name suggests that. Instead, a fifth is a pair of tones that are *five keys apart* on the keyboard. For example, C and G form a perfect fifth. The five keys are C–D–E–F–G. The perfect fifth is foundational in both western and eastern music, and in all musical styles, such as classical, jazz, and pop (Benson, 2006).

One reason why tones with a frequency ratio of 3:2 sound good together is because their waveforms line up. *Figure 9* is an example of how the waveforms might look. You can see how both waveforms start in the same place and end in the same place. The black one is two complete cycles long; the red one is three complete cycles long (Wright, 2009). Thus the 3:2 frequency ratio.

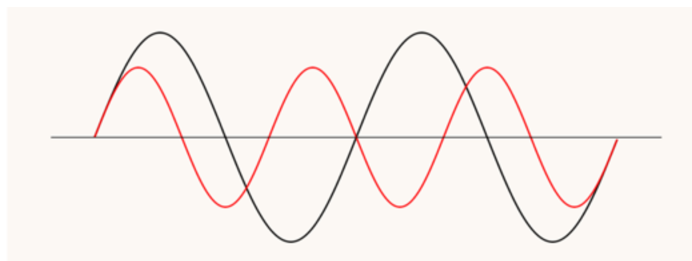

Figure 9: waveforms with 3:2 ratio. Black: $y = sin\,2x$*; Red:* $y = sin\,3x$*.*

Since these waveforms line up, maybe we can use their frequencies to make the most melodious scale possible. This is what Pythagoras tried to do. Suppose we start with A_2 = 110 Hz. Then we can make a perfect fifth by having the next above E tone be 440 x 3/2 = 165. Then we could make another perfect fifth by having the next B tone above that be 660 x 3/2 = 247.5. We could continue doing this 12 times to determine all the frequencies of all the 12 tones.

We can visualize this process as a spiral, similar to the spiral we saw earlier, but instead of using adjacent keys on the piano keyboard, using fifths (Benson, 2006). *Figure 10* shows such a spiral in a two-dimensional space.

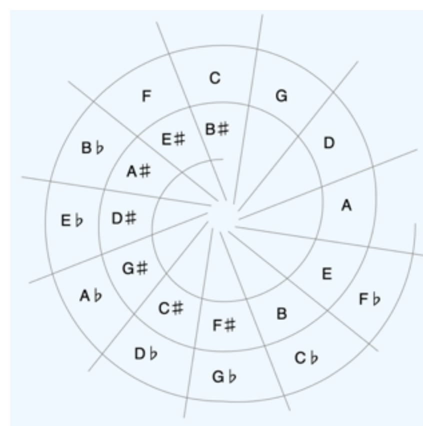

Figure 10: A spiral of fifths

In the end, we would have a set of frequencies that looks like this:

A	110	C♯	556.875	F	2819.179688
E	165	G♯	835.3125	C	4228.769531
B	247.5	D♯	1252.96875	G	6343.154297
F♯	371.25	A♯	1879.453125	D	9514.731445
				A	14272.09717

Note that the first A is 110, and the last A is 14272.09717. These A tones are 7 octaves apart. Since an octave has a frequency ratio of 2:1, we would expect that last A to have a frequency of $110 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ or 110×2^7 = 14080. So there is a little discrepancy at the end. The discrepancy, expressed as a ratio, is 14272.09717 / 14080 = 1.0136432649. This ratio is called the *Pythagorean Comma* and is the error created when trying to tune a piano this way (Wright, 2009). It is not possible to resolve this discrepancy because $(3/2)^{12}$ and 2^7 have no factors in common. The history of tuning revolves around this fundamental problem and finding ways to resolve it. The spiral is one way to visualize this problem.

Figure 11 shows these Pythagorean fifths in a spiral, similar to the spiral we saw previously for scales. In this case, the adjacent balls represent fifths, not adjacent keys on the piano. As we move higher on the spiral, we increment by fifths. But after we've moved around 12 fifths, the ball does not quite line up with the ball above it. This represents the 1.0136 discrepancy.

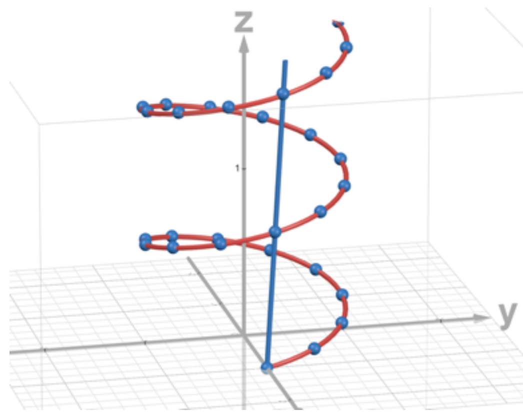

Figure 11: Pythagorean fifths in a spiral

Equal-tempered tuning, developed in the 1700s, is the modern solution to this problem (Benson, 2006). The mathematical strategy is simple. We start with the premise that a 2:1 frequency ratio is an octave. There is a 2:1 frequency ratio between A_3 and A_4, and a 2:1 ratio between A_4 and A_5. If A_4 is 440 Hz, then A_3 is 220 Hz, and A_5 is 880 Hz. The relationship is geometric, not linear. So we get the next frequency in the series by multiplying by a fixed number. In the case of an octave, that number is two. For adjacent notes in the scale, since there are 12 notes in an octave, we'd need to multiply 12 times by a fixed number to arrive at 2. So that fixed number would be $\sqrt[12]{2} \approx 1.05946$ (DuBose-Schmidt, 2022). Each tone has a frequency that is $\sqrt[12]{2}$ times the frequency of the next lower tone.

Since this does not result in small integer ratios for the various intervals, the waveforms do not "line up" the way we saw in *Figure 9* with the 3:2 waveforms. For example, the ratio of a perfect fifth becomes 1.4983, which is pretty close to 1.5. This is close enough for most practical purposes when performing music. But some composers use special scales, instruments, and performance styles that are capable of preserving the small integer ratios, and they can sound different.

3. Chords

When we talk about chords, we are usually talking about triads. A triad is three tones played at the same time. The most common chords are the major triad and the minor triad. Both triads have the root of the chord and the perfect fifth. The *root* is the basis of the chord. For example, in the key of C, the root of the chord would be the note C. We can then add the perfect fifth, which is G. In between, there will be another tone.

If the chord is a C *major chord*, the additional tone will be E, which is a *major third* above the root, C. Like the perfect fifth, the third is not a fraction, but takes its name because the notes are three white keys apart. Looking at the piano keyboard again, the E key is four keys above the C key, if we count all the white and black keys. C–C♯–D–D♯–E. The major third interval is always four keys distance.

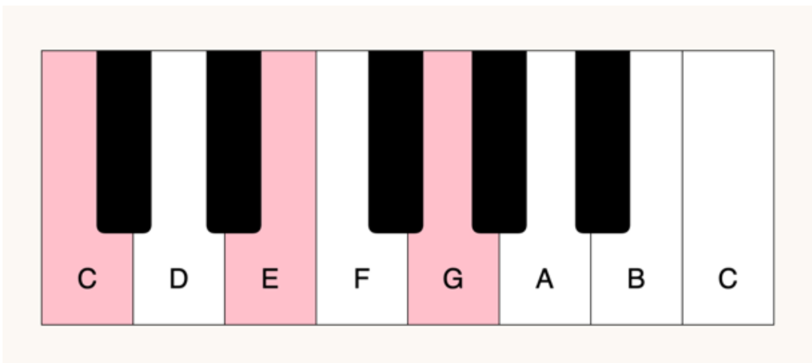

Figure 12: Major Chord, C–E–G

To make a *minor chord*, the additional tone will be a *minor third* above the root note. A minor third is a distance of three keys. In the case of the C chord, the minor third will be the E♭ key, which is a distance of three keys above the C key.

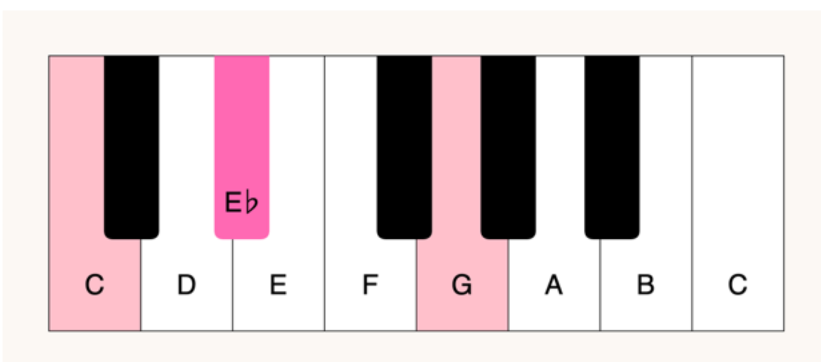

Figure 13: Minor Chord, C–E♭–G

The C major chord consists of a major third interval (C to E) and a minor third interval (E to G). The C minor chord consists of a minor third interval (C to E♭) and a major third interval (E♭ to G). Both chords have the same intervals, major third and minor third, but in a different order. The quality and mood of the two chords is different. Many people think the major chords invoke a happier mood and the minor chords a more melancholy mood (Loveday 2022).

Chords can have more than three tones, but usually we think of such chords as being variations on the three-tone chords. An example is the *seventh chord* used in all kinds of western music, but especially in jazz music. A seventh chord might consist of the tones C–E–G–B♭ or G–B–D–F. Note the addition of the fourth tone to the major triad. In each case, the additional tone is seven white keys above the root.

Euler (1739) posited placing pitch classes in equilateral triangles in the plane. From any pitch, movement to the right goes up in fifths, movement up and to the right goes up in major thirds, and movement down and to the right goes up in minor thirds. The layout is called a *Tonnetz* (tone net).

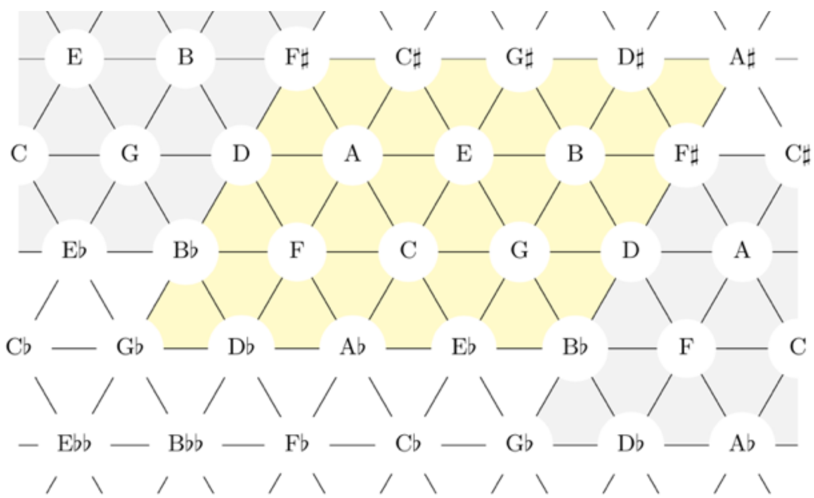

Figure 14: Euler Tonnetz (Rietsch, 2024)

In this mesh, triangles pointing upward are major triads, and triangles pointing downward are minor triads. Note that the left edge of the yellow shaded portion has the same pitch classes as the right edge. This allows us to extend the geometry indefinitely or even wrap it around, forming a torus (Rietsch, 2024). The top and bottom edges, while their tones are enharmonic (different symbols for the same tone), may not perfectly align, depending on the particular method of tuning being used. In the modern equal-tempered tuning system, G♭ and F♯ are the same note, as are D♭ and C♯, and so on. But in other systems of tuning, such as Pythagorean or Just Intonation, these tones may not be the same.

Eitz utilized Euler's triangular array to notate different options of just intonation (Benson, 2006). Eitz added numbers to indicate the correction in commas from the idealized Pythagorean tuning. *Figures 15* and *16* show Eitz's notation.

Figure 15: Eitz notation for major triad (left) and minor triad (right). (Benson, 2006)

A major triad is a triangle pointing up. E^{-1} in the major triad signifies that the third (E) is being adjusted down by one comma (a factor of 81/80 in frequency).

A minor triad is a triangle pointing down. $E\flat^{+1}$ signifies that the minor third (E♭) is being adjusted upward by one comma in frequency. This notation allows us to succinctly notate an entire Just tuning. An example is Kepler's tuning (Benson, 2006):

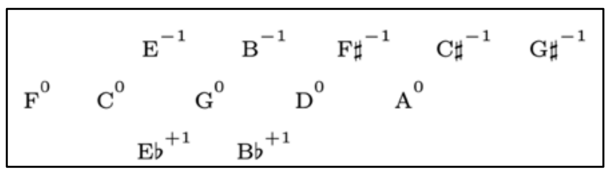

Figure 16: Kepler's just tuning using Eitz notation. (Benson, 2006)

When we draw the entire scale, it resembles Euler's Tonnetz.

Rietsch (2024) suggests other triangular and polygonal meshes that provide more uses, for example, describing chords with four or more tones, such as the seventh.

Conclusion

We have looked at various ways to visualize music. We can view tones as keys on a keyboard or a circle of tone classes or a spiral of frequencies. We can view scales as waveforms or as spirals. We have seen how the notes of the scale form a geometric series of frequencies. We have seen how chords can be viewed as a mesh or grid that shows the relationships between minor thirds, major thirds, and perfect fifths.

There are many other ways to visualize music using geometry. The Euler Tonnetz can be bent and folded in various ways to create an infinite continuous strip or even a torus (Shepard, 1982). These attempts aim to harmonize the physical attributes of frequency and timbre with their psychological perception.

The (Hugo) Riemann Wreath (*Figure 18*) shows possibilities of chord progressions (changes of harmony over time) (Morris, 1998). The wreath is useful for modern and jazz composers who are attempting to create new musical progressions.

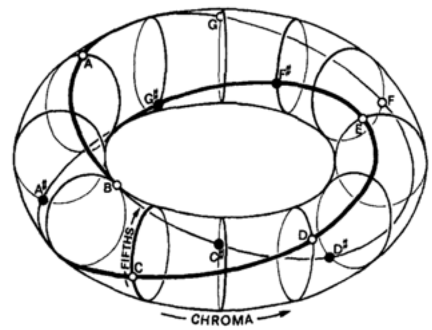

Figure 17: Torus (Shepard, 1982)

There are ways of visualizing rhythm using marks on circles (similar to our tone circles and spirals) to indicate beats within a measure (Demaine, 2009). These geometric constructs enable researchers to understand world music and perhaps explore common origins among different musical cultures (*Figure 19*).

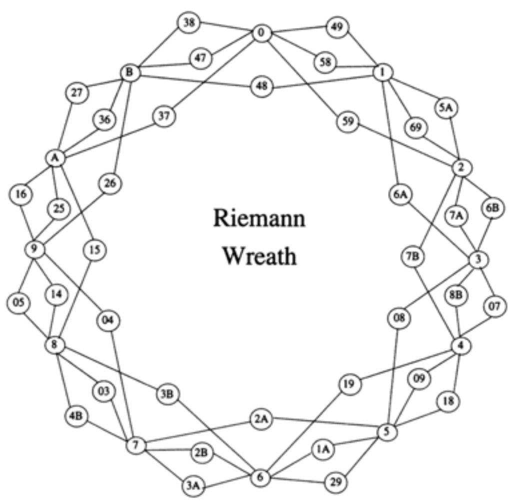

Figure 18: Riemann Wreath (Morris, 1998)

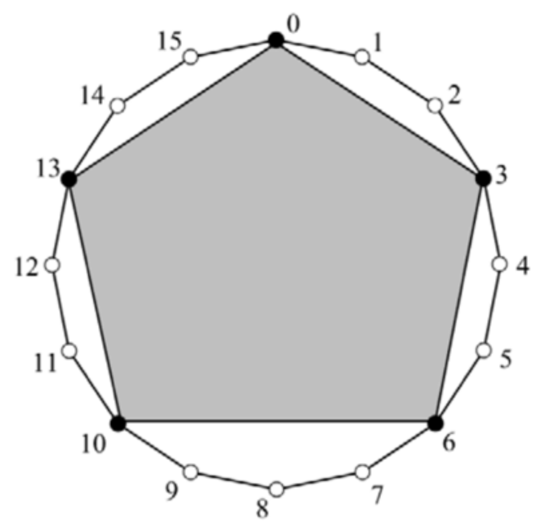

Figure 19: Bossa nova rhythm (Demaine, 2009)

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