Theorem Inverse and Converse

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Introduction

Theorems. In this article we will discuss theorems and how to use them. A *theorem* is a statement that we can prove. We prove the theorem by referring to fundamental statements we call *axioms*, *postulates*, or just assumptions. We don't need to prove the axioms because we assume them to be true based on our observations of the world.

Axioms. Example of axioms are: (1) We can draw a straight line between any two points. (2) We can describe a circle in terms of its center and radius. (3) All right angles are equal to one another. There are anywhere between around 10 and 30 axioms, depending on how we count them. We will use a few of them here.

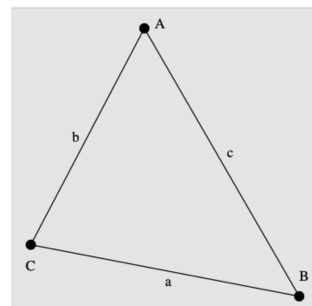

Figure 1: Triangle with angles ABC, sides abc

Triangle. We will talk about the geometric shape known as a triangle.
A *triangle* is a shape that has three *sides* and three corners, called *angles*. You are probably already familiar with triangles.

Congruence. We will talk about the concept of *congruence*. Congruence is a big word that just means two things are the same. Two lines are said to be *congruent* if they have the same length. Two angles are congruent if they have the same measure (which you might know of as degrees). Two triangles are congruent if all their corresponding sides and angles are congruent.

Opposite sides and angles. Figure 1 shows a triangle ABC. The angles are A, B, and C. The sides are a, b, and c. Side a is opposite angle A. Side b is opposite angle B. Side c is opposite side C. In a triangle, the side opposite an angle is the side that doesn't touch that angle. The angle opposite a side is the angle that doesn't touch that side.

Converse and Inverse. In this article we will discuss the *converse* of a *theorem* and the *inverse* of a theorem. These are new statements that may be useful for further study, but we'd have to prove them first. We will look at the *Isosceles Triangle Theorem* and try to see if its converse and inverse are true.

Contradiction. A contradiction is two statements that cannot both be true at the same time. For example, $5 < 10$ and $5 > 10$ are statements that contradict each other. $A = B$ and $A \neq B$ are statements that contradict each other.

The purpose of this article is that the student will recognize a theorem and related statements such as the converse of the theorem and the inverse of the theorem. A theorem can be proven true using things that we already know are true or false. Just because a theorem can be proven true, doesn't mean that the converse or the inverse are true. We will see examples how these statements work.

The Isosceles Triangle Theorem

The Isosceles Triangle Theorem. This theorem says, "If two sides of a triangle are congruent, then the angles opposite those sides are congruent." [1]

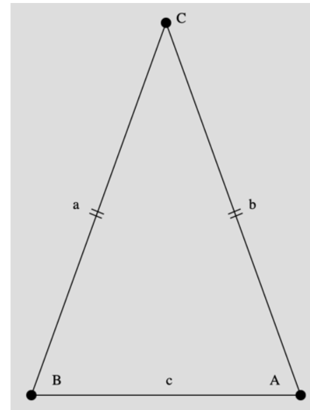

Figure 2: Isosceles Triangle

An *isosceles triangle* has two sides that are congruent, or have the same length. In Figure 2, sides a and b are congruent. The two little lines mean that the two sides are congruent. The theorem says since sides a and b are congruent, then angles A and B are also congruent, or have the same measure. It sure looks like it, but we should prove the theorem.

This theorem concerns itself with the question, is it always the case that the two angles will be the same? We drew a triangle that looks like it, but can we show that this is true for any shape isosceles triangle?

Suppose we split the triangle down the middle by splitting angle C exactly in half, forming two triangles that are reflections of each other. Since we have split angle C exactly in half, the two half-angles are congruent, that is, they have the same measure. We already know that sides a and b are congruent. And the new side d is congruent to itself. Are the two half-triangles congruent to each other?

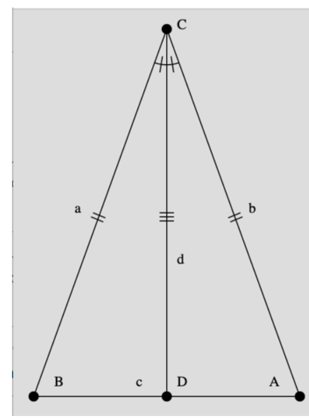

Figure 3: Split the triangle

Fortunately, we have an axiom that helps us decide. The *SAS Postulate* (or *SAS Axiom*) says that if we have two triangles where two of the sides are congruent and the angle between the two sides is congruent, then the triangles are congruent. (SAS means side-angle-side.) Side a is congruent to side b. Side d is congruent to itself. The angle between them is the half angle we created by splitting angle C. So by the SAS Postulate, the two half-triangles are congruent.

Remember that if two triangles are congruent, all their corresponding sides and angles are congruent. In particular, angle B in the left half-triangle is the corresponding angle to angle A in the right half-triangle. Since the corresponding angles are congruent, we know that angles A and B are congruent. These are the angles opposite the congruent sides a and b, so we have proved the theorem.

Note that the theorem has an "if" part and a "then" part. ***If*** two sides of a triangle are congruent, ***then*** the angles opposite those sides are congruent. These two parts of the statement will be important when we discuss the converse and the inverse.

The Converse of the Theorem

The Converse of the Isosceles Triangle Theorem. "If two angles of a triangle are congruent, then the sides opposite those angles are congruent." [1]

Note that the converse of the theorem has the ***if*** and ***then*** parts switched around. The following chart contrasts the two different statements, piece by piece:

Isosceles Triangle Theorem	Converse
If two sides of a triangle are congruent,	If two angles of a triangle are congruent,
then the angles opposite those sides are congruent.	then the sides opposite those angles are congruent.

See how the theorem says "If two sides … then the angles …" The converse says, "If two angles … then the sides …" It may not be obvious, but these are not the same two statements at all. Do you think the converse statement is true? How would you go about showing that it is true? We could attempt to prove the converse as a theorem.

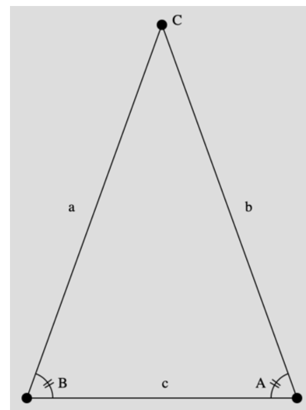

Figure 4: Two Angles the same

Figure 4 shows a triangle that has two angles the same. Angles A and B are congruent to each other. However, we don't know anything about the sides and the other angle. Can we show that sides a and b are congruent?

We can use a strategy similar to what we did before. We can split the triangle by splitting angle C in half. In this case, we know that angles A and B are congruent, but we don't know anything about sides a and b.

See Figure 5. After splitting the triangle in half, we know that the two half-angles at C are congruent, and we know that side d is congruent to itself. So we know that two angles are congruent, and one side is congruent. Is this enough?

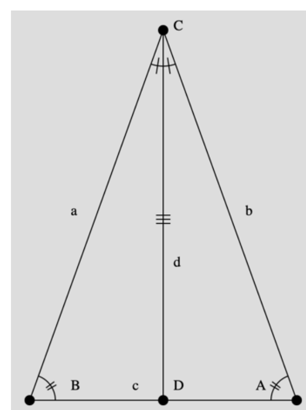

Figure 5: Split the triangle

Fortunately, we have a theorem called the *AAS theorem* that says if two triangles have two angles congruent, and one non-included side congruent, then the triangles are congruent. (AAS means angle-angle-side.) In this case, the non-included side is d, which is not between angles C and B or between angles C and A.

Like before, if two triangles are congruent, all their corresponding sides and angles are congruent. In particular, side a in the left half-triangle is the corresponding side to side b in the right half-triangle. Since the corresponding sides are congruent, we know that sides a and b are congruent. These are the angles opposite the congruent angles A and B, so we have proved the converse of the theorem.

The Inverse

The Inverse of the Isosceles Triangle Theorem. "If two sides of a triangle are not congruent, then the angles opposite those sides are not congruent." [1]

Note that the inverse inserts the word ***not*** into both the ***if*** and ***then*** parts of the theorem. The following chart contrasts the two different statements, piece by piece.

Isosceles Triangle Theorem	Inverse
If two sides of a triangle are congruent,	If two sides of a triangle are **not** congruent,
then the angles opposite those sides are congruent.	then the angles opposite those sides are **not** congruent.

See how the theorem says "If two sides **are** .. then the angles **are**..." The inverse says, "If the two sides **are not** ... then the angles **are not**..." Do you think the inverse statement is true? Can you think about how you might prove it? Do you think you need to prove it?

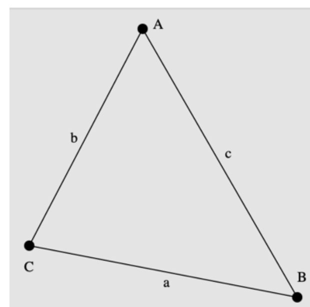

Figure 6: Sides not congruent

One way to think about this kind of a proof is by a technique called *contradiction*. In proof by contradiction, we assume the statement is false, and then show that the argument contradicts itself.

In this example, we are trying to prove that if two sides of a triangle are not congruent, then the angles opposite those sides are not congruent.

We could assume the opposite, that is, the two sides are not congruent, but the angles opposite those sides **are** congruent.

If the two angles are congruent, then the converse of the isosceles triangle theorem says that the sides opposite those angles must also be congruent. But we have already asserted that the two sides are **not** congruent. So we have a contradiction. The sides cannot be both congruent and not congruent at the same time. If we assume the statement is false, we arrive at a contradiction. This means our original statement must be true.

In this case, the converse of the isosceles triangle theorem is true, and the inverse of the theorem is also true. The proof of the inverse uses the converse as part of the proof. Does this tell you something about the relationship between the converse of a theorem and the inverse of a theorem?

Other Examples: True or False?

We have shown that the isosceles triangle theorem is proved, its converse is proved, and its inverse is proved. Do you think it is always the case that if one is true, the others are also true?

Let us look at some other examples.

Rectangle and Square

We can use the example of a rectangle and a square.

Claim: *If a shape is a square, then that shape is a rectangle.* We recognize this claim as true by the definitions of square and rectangle. By definition, a rectangle is a four-sided shape that has four right angles. By definition, a square is a shape that has four right angles and four sides of equal length. If a shape meets the definition of a square, it also meets the definition of a rectangle.

Figure 1: Rectangle

But now consider the converse of this claim: *If a shape is a rectangle, then that shape is a square.* This statement is not necessarily true. While a rectangle may have four right angles, it does not necessarily have four sides of equal length. So the statement is not always true. In Figure 2, you can see a rectangle that is not a square. So we say the converse is not true. In this case, the fact that the original statement is true does not ensure that the converse is also true.

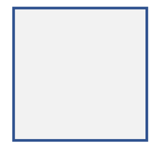

Figure 2: Square

The original claim: *If a shape is a square, then that shape is a rectangle.* We recognize this statement as true by virtue of the definitions, and we gave a rationale for believing this is a true claim.

But now consider the inverse of this claim: *If a shape is not a square, then that shape is not a rectangle.* This statement is not necessarily true. In Figure 2 above, you can see a shape that is not a square, but it is a rectangle. So this example demonstrates that the inverse of the original claim is false. The fact that the original claim is true obviously does not ensure that the inverse is also true.

Number Comparisons

Claim: If a number is less than 5, then it is also less than 10. We can see that this is true. Any number less than 5 is also less than 10.

But consider the converse of this claim. If a number is less than 10, then it is also less than 5. We can easily show this is not true by looking at the number 7. 7 is less than 10, but it is not less than 5. So the converse statement is not true.

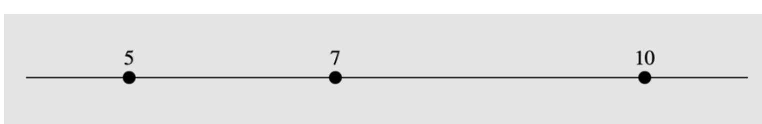

Figure 3: Number line showing 5 < 7 and 7 < 10

Also consider the inverse of this claim. If a number is not less than 5, then it is not less than 10. Again, consider the number 7. 7 is not less than 5. But it is also **not not** less than 10; it is in fact less than 10. (Don't worry if this sounds strange. Statements of the form **not not** are hard to think about.)

Uses of Isosceles Triangles

Using knowledge gained in studying isosceles triangles, we can learn to bisect line segments using compass and ruler. We can also learn to bisect angles using compass and ruler. We can find the exact center of a triangle, which is useful in balancing it.

Isosceles triangles have many uses in real life. Some electric guitars have this shape, which makes them easy to play. Even closet hangers have the shape of an isosceles triangle. Isosceles triangles are used in origami paper folding because they have a balanced look. Thinking bigger, isosceles triangles are used in architecture and engineering. They allow weight to be distributed properly on a roof. They are frequently used when building bridges.

Conclusion

In this article, we have discussed the concept of a *theorem*, which is a claim we attempt to prove true. We also discussed the related claims called the *inverse* and *converse* of the theorem. These claims look similar in some ways to the original theorem, but just because we prove the theorem true, this does not mean the related claims are also true. If we wanted to prove them true, we would need to prove them separately. Sometimes we can give counter examples that show the inverse and the converse are not true. Sometimes the claims are true, and sometimes they are not.

Footnotes

[1] CSU East Bay Class Notes, Math 396W.

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Note: The Flesch-Kincaid score says this is suitable for 6th grade and calls it easy to read.
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