Hyperbolic Geometry Using the Hyperbolic Half Plane Model

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Introduction

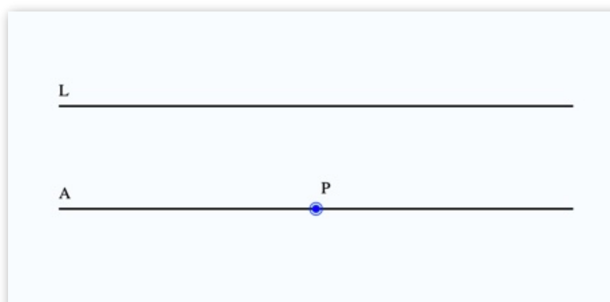
Most of what we know as geometry originated with Euclid's book *Elements*, written around 300 BC. This book contains a series of *axioms* or *postulates*, which are statements about geometry or math that we take for granted because they are considered to be obvious. Some examples of postulates, from a modern language version (School Mathematics Study Group, 1961), are:

- Postulate 1. Given two points, there can be only one line that goes through them.
- Postulate 2. Given two points, we can assign them a number that we call the *distance* between them.
- Postulate 11. Given an angle we can assign it to number called its measurement in degrees, for example 30°.
- Postulate 15. Two triangles are the same, or *congruent*, if they have two sides the same and the angle between those two sides is the same.

When we study Euclidean geometry, we are usually looking at objects in the two-dimensional plane. Such geometric objects are things that we can draw on a piece of paper: line segments, angles, and polygons such as triangles.

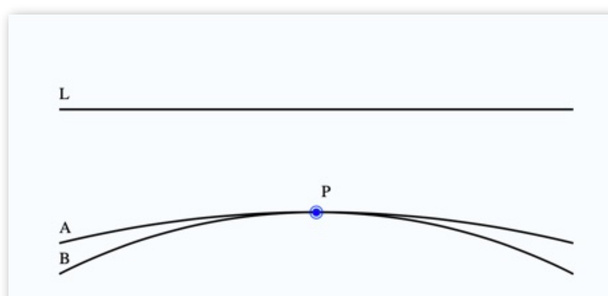
Parallel Lines

Postulate 16, the Parallel Postulate, says that given a line and a point not on that line, there can be only one line is parallel to that line. Like all postulates, this seems obvious. Line A below is the only line parallel to line L that goes through point P.



Line A is the only line through point P and parallel to line L.

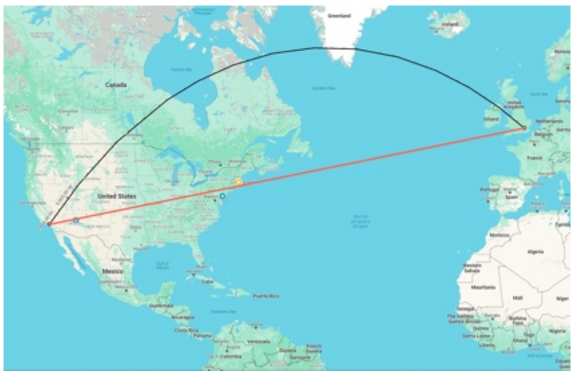
But mathematicians have sometimes wondered, could there be a world where there are multiple lines parallel to the first line? A geometry where there can be multiple lines through a single point, all parallel to another line, might look like this, where neither of the lines through point P intersect with line L:



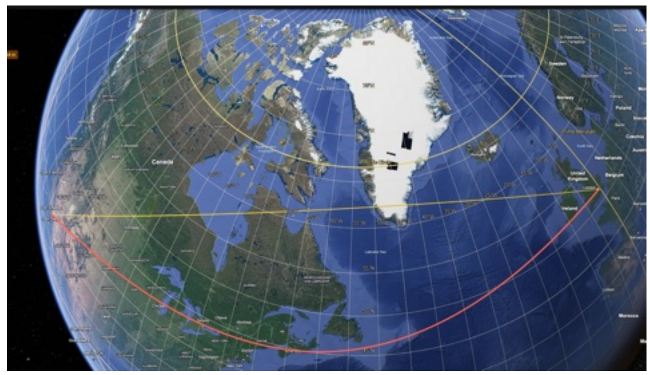
Lines A and B are both parallel to line L.

But aren't those lines curved? They don't look like lines. But maybe it's not the lines that are distorted, but the way we are drawing them. What if lines A and B really are parallel to L and really are straight but it's just the way we are drawing them that makes them look curved?

An example comes from the globe. You may have wondered why airplanes fly over Greenland when they could just fly "straight" to Europe or Asia. For example, the following Mercator Projection map shows the path to fly from Los Angeles to London. Note that the path looks curved. But it is actually straight; the curvature of the earth makes it look like a longer path. The flatness of the two-dimensional map makes it look longer, but it is not. On the actual globe, this is the shortest-distance path from North America to Europe.



On the map, the orange line looks shorter but the black line is actually the shortest path. (Reddit, 2024)

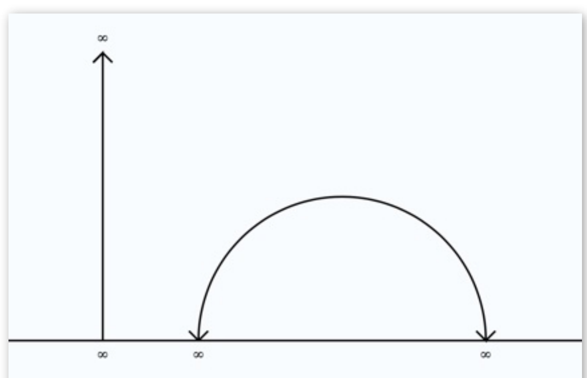


On the globe, the yellow line looks shorter because it is actually the shortest path. (Reddit, 2024)

Hyperbolic Geometry

Hyperbolic geometry is another kind of geometry where things that look straight may not actually be straight and things that look curved may actually be the shorter path. The specific model we'll look at is called the *Hyperbolic Half Plane* model. We call it the Hyperbolic Half Plane because it uses only the first and second quadrants of our usual Cartesian plane. That is, the half plane where $y \geq 0$ and x can take any value.

In Euclidean geometry, lines as go off to infinity in both directions. On the Hyperbolic Half Plane, the x axis is considered infinity, and all lines go to infinity. Because the x axis is considered infinity, lines can go infinitely upwards and down to the x axis. But other lines could have both ends going down to the x axis. We draw this second kind of lines as semicircles. Here are examples of these two kinds of lines.

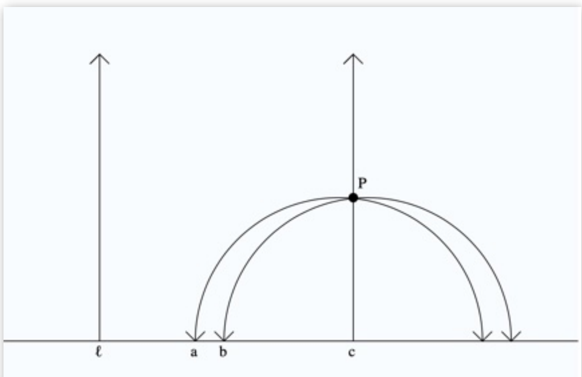


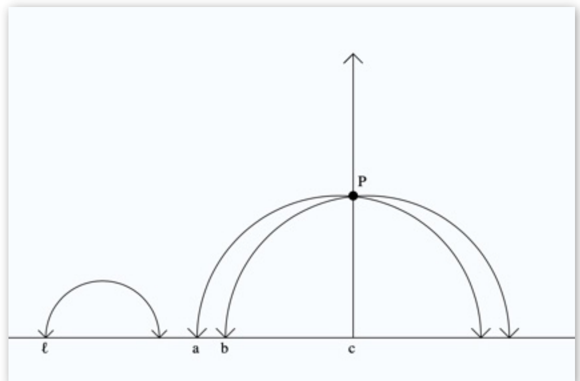
The line on the left starts at infinity (the x axis) and goes up to infinity. The line on the right is a semicircle with its center on the x axis and both ends go to infinity (the x axis).

A consequence of the x axis being infinity is that objects closer to the x axis are larger than they appear. Two objects may appear to be the same size, but the one closer to the x axis is larger. Two objects may actually be the same size in hyperbolic space, but the one farther from the x axis will appear larger.

Parallel Lines

In the hyperbolic half plane model, we might have multiple lines, all going through the same point, and all parallel to another given line. Our usual Euclidean geometry does not allow this. But the Hyperbolic Half Plane geometric model does allow this. In the drawings below, lines a, b, and c are all parallel to line ℓ. All the parallel lines go through point P.



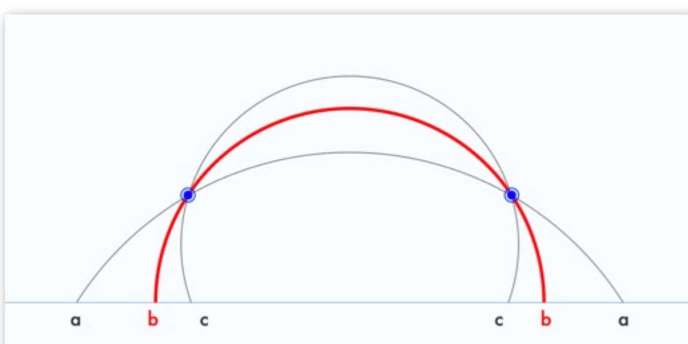


Neutral Geometry

A geometry that satisfies the first 15 Euclidean postulates, ignoring the Parallel Postulate, is called a *Neutral Geometry*. The Hyperbolic Half Plane model is a Neutral Geometry. This means it satisfies the first 15 postulates. Euclidean geometry is also a Neutral geometry, because it satisfies the first 15 postulates, plus the Parallel Postulate (#16).

Lines and Segments

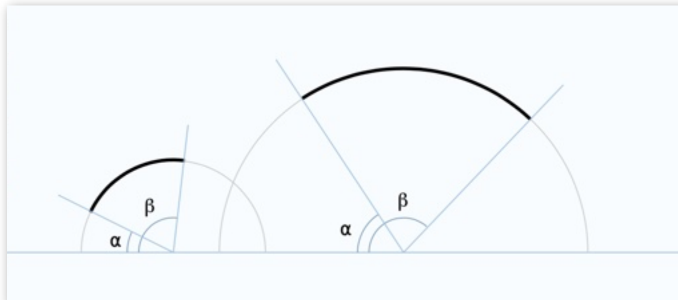
Postulate 1. Given two points, there can be only one line that goes through them.



The red circle (b) is the only line through the two blue points. The other circles do not have centers on the x axis. You might think that circle a travels the shorter distance between the two points, but that segment lies closer to infinity, so it actually travels a longer distance. This is similar to the flight path where the line that looks shorter on the Mercator map is actually longer than the true distance.

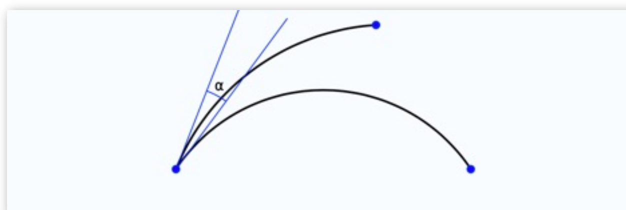
Postulate 2. Given two points, we can assign them a number that we call the *distance* between them. In the hyperbolic half plane model, the distance between two points has the equation

$$length = \ln \frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha}$$



Where $\alpha < \beta$ and α and β are the angles formed by the radii of the circle at the endpoints. In the diagram, the two line segments (arcs) have the same length. The one on the left looks larger because it is lower, therefore closer to infinity. Also $\beta - \alpha$ is not the same for both segments, because the one on the right is more centered, while the one on the left lies more leftward on its defining circle.

Angles



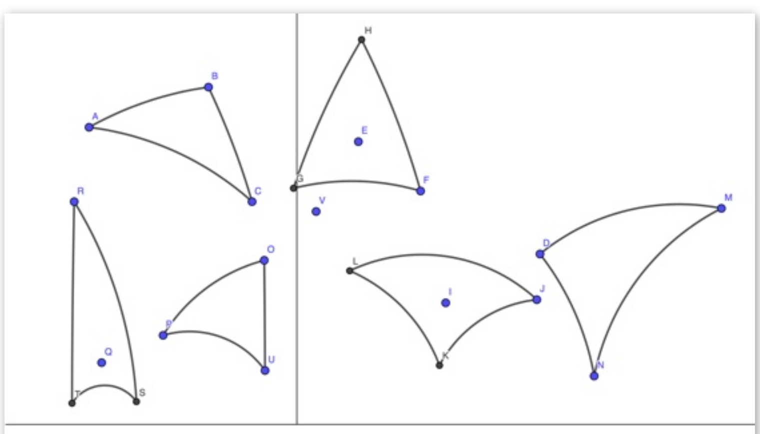
An *angle* consists of two different lines or segments joining at the same point, similar to Euclidean geometry.

Postulate 11. Given an angle we can assign it to number called its measurement in degrees. The measure of the angle is the same as the angle between the tangent lines.

We can use segments and angles to make polygons such as triangles.

Triangles

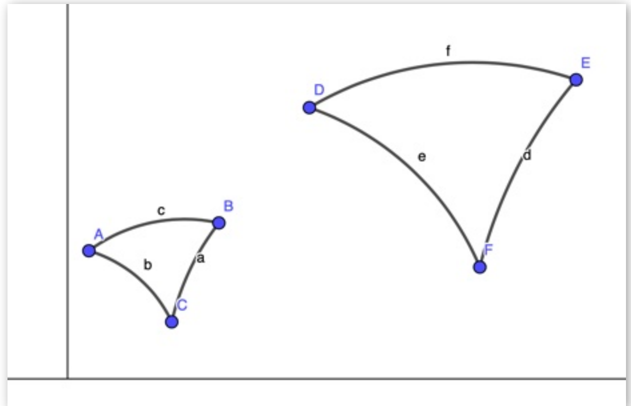
The Hyperbolic Half Plane model also has triangles, but you may think they look a bit odd:



Each triangle has three line segments, similar to Euclidean geometry. The segments that compose each triangle are the shortest paths between the triangle points. Each triangle edge is either a straight line up and down or a piece of a circle whose center is on the x axis. Note that the segments closer to the x axis look more curved, because they have a smaller radius. Segments farther from the x axis look less curved, because they have a larger radius.

Postulate 15. Two triangles are the same, or *congruent*, if they have two sides the same and the angle between those two sides is the same. The following two triangles are congruent, even though they might not look like they are the same size.

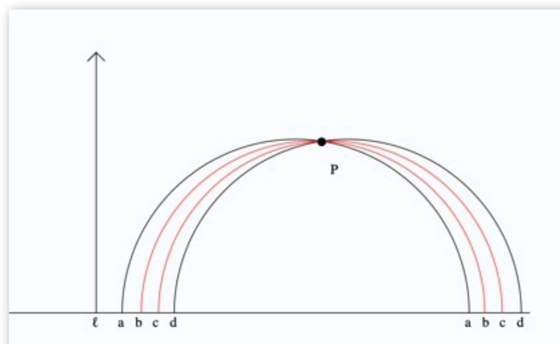
The triangle on the right looks larger because it is farther away from infinity. But the side and angle measures are exactly the same in the hyperbolic space.



Congruent triangles.

Questions

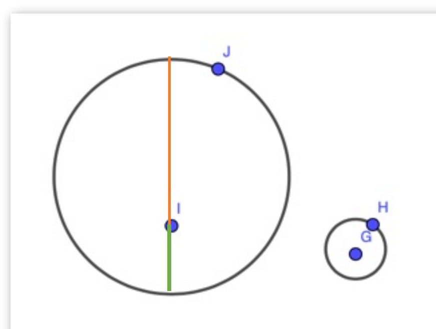
Having established that hyperbolic geometry allows for multiple parallel lines through a point, a natural question is, how many parallel lines could there be? Could there be an infinite number of parallel lines? One way to visualize this is to think about how many lines there might be *between* two other lines. The drawing shows two lines a and d parallel to ℓ. Lines b and c are "between" lines a and d. What about other possible lines between a and d? How many lines are there "between" a and d? For every line we find in between, we can find another line in between the old ones and the new ones created, for example, between lines a and b. This suggests there may be an infinite number of parallel lines. But maybe there are some cases where this doesn't hold.



Infinite number of parallel lines?

What about circles? It turns out we can draw circles in the hyperbolic space, and they look like regular circles. But the centers may not be where we expect them to be.

When the circle is smaller, as in the one on the right, the center looks close to where we might expect it to be. But when the circle is larger, as in the one on the left, the center appears far away from where we might expect. This is because the part of the circle below the center is closer to infinity so a small vertical distance is actually a larger distance in hyperbolic space.



Circles in the hyperbolic half plane.

The part of the circle above the center is farther from infinity, so the distances are smaller. The orange radius has the same length as the green radius.

What about other conic sections? Ellipse? Parabola? Do these exist in the hyperbolic half plane model and what do they look like?

Conclusion

The hyperbolic half plane model shows us a kind of geometry that obeys most of the postulates of classical geometry, but has the additional capability of multiple parallel lines to a line L through a point P, which is not possible in Euclidean geometry.

Hyperbolic geometry has the familiar objects such as lines, line segments, segments lengths, angles, angle measurement (such as degrees), triangles, and congruent objects, although these look different; in particular, lines and line segments have curvature. Parallel lines behave differently in hyperbolic geometry than we may be used to from studying Euclidean geometry. Segments and triangles may appear to be different sizes yet still be congruent.

References

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